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L_∞-Boundedness of L_2-Projections on Splines for a Geometric Mesh

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In this paper we complete the investigations started by K. Höllig and K. Scherer "Approximation Theory, III" (E. W. Cheney, Ed.), Academic Press, New York, 1980. We study C. de Boor's conjecture of the L_∞-boundedness of the L_2-projection P on smooth splines in the special case of a geometric mesh $\mathbf{x} = \{q^\nu\}$. A connection to the interpolation projection is established and the uniform boundedness of $\|P\|_\infty$ with respect to q is proved.

1. INTRODUCTION

For a biinfinite strictly increasing sequence of knots $\mathbf{x} = \{x_\nu\}_{\nu \in Z}$ we denote by $N_{k,\nu}$, $\operatorname{supp} N_{k,\nu} = [x_\nu, x_{\nu+k}]$, the B-splines of order k which form a partition of unity on $(x_{-\infty}, x_\infty)$ [5]. $S_{k,\mathbf{x}} = \operatorname{span}\{N_{k,\nu}\}$ is the corresponding space of splines.

If the matrices

$$\begin{aligned}(A_{k,r})_{\nu\mu} &= N_{2k,\mu}(x_\nu), && r = 0 \\ &= \frac{r}{x_{\nu+r} - x_\nu}\,(N_{r,\nu}, N_{2k-r,\mu}), && 0 < r < 2k\end{aligned} \tag{1}$$

are invertible on l_∞, we can define the projections $P_{k,r}\colon C(x_{-\infty}, x_\infty) \to S_{2k-r,\mathbf{x}}$ by the linear systems

$$\begin{aligned}P_{k,r} f &= \sum_\nu a_\nu(f)\, N_{2k-r,\nu} \in L_\infty \cap S_{2k-r,\mathbf{x}}, \\ \sum_\mu (N_{r,\nu}, N_{2k-r,\mu})\, a_\mu(f) &= (N_{r,\nu}, f).\end{aligned} \tag{2}$$

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Hence $P_{k,0}$ is the interpolation projection and $P_{k,k}$ the usual L_2-projection. Using the well known estimates [5]

$$C_k \|\{a_v\}\|_{l_p} \leqslant \left\| \sum_v a_v \left(\frac{k}{x_{v+k}-x_v}\right)^{1/p} N_{k,v} \right\|_p \leqslant \|\{a_v\}\|_{l_p}, \tag{3}$$

where C_k is a positive constant, we see that [cf. 4]

$$C_k^2 \|(A_{k,r})^{-1}\|_\infty \leqslant \|P_{k,r}\|_\infty \leqslant \|(A_{k,r})^{-1}\|_\infty . \tag{4}$$

For bounded global mesh ratio, the boundedness of the projections $P_{k,0}$ and $P_{k,k}$ with respect to the L_∞-norm was shown by C. de Boor in [6, 7, cf. also 12]. He conjectured [3] that for the L_2-projection $P_{k,k}$ this result remains valid for an arbitrary sequence of knots and $\|P_{k,k}\|_\infty$ can be bounded by a constant which only depends on k and not on the mesh. In general, however, the matrices $A_{k,r}$ need not to be invertible. In fact, C. A. Micchelli showed in [16] that for a geometric mesh

$$\mathbf{x} = \{q^v\}_{v\in Z}, \qquad q = e^t, \qquad 0 < t < \infty, \tag{5}$$

there are exactly $k-1$ values of q for which interpolation at the knots does not define a bounded projection on $C(x_{-\infty}, x_\infty)$. Therefore a geometric mesh (5) is a reasonable test for de Boor's conjecture. In this case the B-spline basis becomes very simple

$$N_{k,v}(x) = N_{k,0}(q^{-v}x) =: N_k(q^{-v}x) \tag{6}$$

and $A_{k,r}$ is a constant band matrix

$$(A_{k,r})_{v\mu} = \frac{r}{q^r-1}\,(N_r, N_{2k-r}(q^{v-\mu}\cdot)). \tag{7}$$

We have set $N_r(x) = \delta(x)$ and $r/(q^r-1) = 1$ for $r = 0$. Moreover the matrix $A_{k,0}$ is totally positive [8, 15] and therefore, by a standard argument, this is also true for $A_{k,r}$, $0 < r < 2k$. Hence $A_{k,r}$ is invertible on l_∞ iff the characteristic polynomial

$$Q_{k,r}(q; z) = \frac{r}{q^r-1}\, z^{2k-1-r} \sum_v (N_r, N_{2k-r}(q^{-v}\cdot))\, z^v \tag{8}$$

does not vanish for $z = -1$. In this case we have

$$\|P_{k,r}\|_\infty \sim \|(A_{k,r})^{-1}\|_\infty = |Q_{k,r}(q; -1)|^{-1}. \tag{9}$$

We remark that in the case of equidistant knots the corresponding matrices $A_{k,r}$, $0 \leqslant r < 2k$, coincide up to a translation and $(2k-1)!\, Q_{k,r}(1; \cdot)$ equals the Euler–Frobenius polynomial as discovered by Schoenberg [17].

In [14] the following relationship between the characteristic polynomials was shown:

$$Q_{k,r}(q;z) \sim_q Q_{k,0}(q;q^r z), \qquad 0 < r < 2k. \tag{10}$$

Following a suggestion of Scherer we give a new proof of (10) by establishing a convolution formula for the $\mathscr{L}$-splines corresponding to a geometric mesh. This extends the classical result in the equidistant case [17]

$$N_k = N_{k-j} * N_j. \tag{11}$$

Our approach via elementary Fourier analysis is completely analogous to the treatment of the equidistant case and leads to an easy proof of the L_∞-boundedness of $P_{k,r}$ for $r = k-1, k$ and any q. By computing the limits of $Q_{k,r}(q;-1)$ for $q \to \infty$ we prove the uniform boundedness of $\|P_{k,r}\|_\infty$ with respect to $q \in (1, \infty)$ in these cases. For $r \neq k-1, k$, however, there is at least one value of q for which $\|P_{k,r}\|_\infty$ is unbounded. Moreover we extend our results to L_p, $1 \leqslant p \leqslant \infty$. In particular we show that for $r \neq k$ the boundedness of $P_{k,r}$ depends on p.

We remark that using our result given in [14], i.e., Corollary 1 of this paper, Theorems 4, 5 have been independently obtained by de Boor and Micchelli.

2. The Characteristic Polynomials $Q_{k,r}$

With a geometric mesh (5) we may associate the Cardinal $\mathscr{L}$-splines corresponding to the differential operator [16]

$$\mathscr{L}_t(D) = D(D-t)\cdots(D-(k-1)t). \tag{12}$$

The B-spline M_k for $\mathscr{L}_t$ is given by

$$M_k(x) = N_k(e^{tx}). \tag{13}$$

Hence $M_k \subset C^{k-2}$ is a piecewise polynomial of degree $k-1$ in e^{tx} with knots at the integers and support $[0, k]$.

We will need some elementary Fourier analysis [18] and to this end we recall the definition of Fourier transform and convolution.

$$(Ff)(y) = \int_{\mathbb{R}} f(x)\, e^{ixy}\, dx,$$

$$(f * g)(y) = \int_{\mathbb{R}} f(y-x)\, g(x)\, dx.$$

LEMMA 1. *If $f \in L_1$ has compact support then Ff is an entire function and we have for $a \in \mathbb{C}$*

$$(F(e^{a\cdot}f))(y) = (Ff)(y - ia). \tag{14}$$

LEMMA 2. *The Fourier transform of M_k is given by*

$$(FM_k)(y) = C_1(k, q) \prod_{v=0}^{k-1} \frac{e^{vt}e^{iy} - 1}{iy + vt}, \tag{15}$$

where $C_1(k, q) = \prod_{v=1}^{k-1} (vt/(q^v - 1))$.

Proof. Set $Fg_k = \prod_{v=0}^{k-1} (e^{vt}e^{iy} - 1)/(iy + vt))$. We first show that g_k coincides with M_k up to a constant factor. This follows from the uniqueness of the B-spline [15] since:

(1) $$\mathscr{L}_t g_k = F^{-1}\left(\prod_{v=0}^{k-1} (1 - e^{vt}e^{i\cdot})\right) = F^{-1}\left(\sum_{v=0}^{k} a_v e^{iv\cdot}\right) = \sum_{v=0}^{k} a_v \delta(\cdot - v),$$

i.e., g_k has the correct smoothness; and

(2) applying Lemma 1, we see that

$$g_k = g_1 * (e^{t\cdot} g_1) * \cdots * (e^{t(k-1)\cdot} g_1),$$

i.e., supp g_k = supp $M_k = [0, k]$. The constant C_1 can be computed from the identity

$$C_1 \prod_{v=0}^{k-1} \frac{q^{v+1} - 1}{(v+1)t} = \int M_k(x) e^{tx} \, dx = t^{-1} \int N_k(y) \, dy = \frac{q^k - 1}{kt}. \quad \blacksquare$$

Most of our arguments rely on the following simple observation.

LEMMA 3. *For $0 < j < k$ we have*

$$M_k = C_2(k, j, q) M_j * (e^{tj\cdot} M_{k-j}), \tag{16}$$

where

$$C_2(k, j, q) = t \frac{(k-1) \cdots (k-j)}{(j-1)!} (q^{k-j} - 1)^{-1} \prod_{v=1}^{j-1} \frac{q^v - 1}{q^{k-v} - 1}.$$

Proof. We apply the Fourier transform to both sides of the Eq. (16) and obtain, by Lemma 2,

$$C_1(k)\prod_{\nu=0}^{k-1}\frac{e^{\nu t}e^{iy}-1}{iy+\nu t}\doteq C_2(k,j)\,C_1(j)\prod_{\nu=0}^{j-1}\frac{e^{\nu t}e^{iy}-1}{iy+\nu t}$$

$$\times\, C_1(k-j)\prod_{\nu=0}^{k-j-1}\frac{e^{\nu t}e^{jt}e^{iy}-1}{iy+jt+\nu t}.$$

A direct computation shows that $C_2(k,j)=C_1(k)/(C_1(j)\cdot C_1(k-j))$. ∎

Passing to the limit, $q\to 1$, (16) reduces to the well known convolution formula for equidistant splines (11).

LEMMA 4. *The B-spline M_k satisfies the symmetry relation*

$$M_k(k-x)=C_3(k,q)\,e^{-t(k-1)x}M_k(x), \tag{17}$$

where $C_3(k,q)=q^{k(k-1)/2}$.

This again follows from (14) and (15) similar to the proofs of the previous lemmas.

After these preparations we now give an alternative proof of Lemma 4 in [14].

THEOREM 1. *For $0<j<k$ we have*

$$(N_j, N_{k-j}(q^{-\nu}\cdot))=C_4(k,j,q)\,q^{j\nu}N_k(q^{j-\nu}), \tag{18}$$

where

$$C_4(k,j,q)=\frac{(j-1)!}{(k-1)\cdots(k-j)}\,q^{j(1-j)/2}(q^{k-j}-1)\prod_{\nu=1}^{j-1}\frac{q^{k-\nu}-1}{q^{\nu}-1}.$$

Proof.

$$\int N_j(x)\,N_{k-j}(q^{-\nu}x)\,dx$$

$$=t\int M_j(y)\,M_{k-j}(y-\nu)\,e^{ty}\,dy$$

$$=tC_3^{-1}\int M_j(j-y)\,M_{k-j}(y-\nu)\,e^{tjy}\,dy$$

$$=tC_3^{-1}q^{j\nu}\int M_j(j-\nu-x)\,e^{tjx}M_{k-j}(x)\,dx$$

$$=tC_3^{-1}C_2^{-1}q^{j\nu}M_k(j-\nu). \tag{19}$$

A direct computation shows that

$$C_4(k, j, q) = tC_2(k, j, q)^{-1}\, C_3(j, q)^{-1}. \quad \blacksquare$$

As a consequence of this result we obtain the following relationship between the characteristic polynomials $Q_{k,r}$ defined by (8).

COROLLARY 1. *The characteristic polynomials* $Q_{k,r}$, $0 < r < 2k$, *are related by*

$$Q_{k,r}(q; z) = C_5(k, r, q)\, q^{r(1-2k)} Q_{k,0}(q; q^r z)$$

$$= C_5(k, r, q)\, z^{2k-1} \sum_{\nu} M_{2k}(-\nu)\, q^{r\nu} z^{\nu}, \tag{19}$$

where

$$C_5(k, r, q) = \frac{r!}{(2k-1)\cdots(2k-r)}\, q^{r(r+1)/2} \prod_{\nu=1}^{r} \frac{q^{2k-\nu}-1}{q^{\nu}-1}.$$

Proof. Applying (18) we obtain by the definition of the polynomials $Q_{k,r}$

$$Q_{k,r}(q; z) = \frac{r}{q^r - 1}\, C_4(2k, r, q)\, z^{2k-1-r} \sum_{\nu} M_{2k}(r-\nu)\, q^{r\nu} z^{\nu}$$

$$= \frac{r}{q^r - 1}\, C_4 q^{r^2} z^{2k-1} \sum_{\mu} M_{2k}(-\mu)\, q^{r\mu} z^{\mu}. \quad \blacksquare$$

Remark. Since the zeros of $Q_{k,r}(q; \cdot)$ are distinct, Corollary 1 can be viewed as a relation between the zeros $\lambda_\nu(q)$ of $Q_{k,0}(q; \cdot)$ and $\lambda_{\nu,r}(q)$ of $Q_{k,r}(q; \cdot)$, viz.,

$$\lambda_{\nu,r}(q) = \lambda_\nu(q)/q^r, \qquad \nu = 1,..., 2k-2 \tag{19$'$}$$

which extends the well known equality $\lambda_{\nu,r}(1) = \lambda_\nu(1)$. Our proof was based on generalizations (15)–(18) of identities for the equidistant case which we believe to be of independent interest. As pointed out by de Boor, (19′) can be also proved directly and we include the nice argument.

Suppose λ is a zero of $Q_{k,0}(q; \cdot)$, i.e., $\sum_\nu \lambda^\nu N_{2k,\nu}$ is a nullspline for the interpolation problem. Taking divided differences, it follows by the Peano kernel theorem that

$$|q^\mu,..., q^{\mu+r}| \sum_{\nu} \lambda^\nu N_{2k,\nu} = \Big(N_{r,\mu}, D^r \sum_{\nu} \lambda^\nu N_{2k,\nu}\Big) = 0.$$

Applying successively the formula for differentiating a B-spline expansion

[5] and making use of the special structure of a geometric mesh, we end up with

$$F(q;\lambda)\sum_{\nu}(\lambda/q^{r})^{\nu}(N_{r,\mu},N_{2k-r,\nu})=0.$$

The constant

$$F(q;\lambda)=\prod_{j=1}^{r}\frac{(2k-j)(1-q^{j-1}/\lambda)}{q^{2k-j}-1}$$

does not vanish for the negative zeros λ of $Q_{k,0}(q;\cdot)$. Hence we conclude that λ/q^{r} is a zero of $Q_{k,r}(q;\cdot)$.

The polynomial $Q_{k,0}$ was investigated in [16] in connection with the interpolation problem for Cardinal $\mathscr{L}$-splines. In particular the following formula for $Q_{k,0}$ was obtained (with different normalization).

THEOREM 2 [16, (23)]. *For $q=e^{t}$ and $r\in\mathbb{R}$ we have*

$$Q_{k,0}(q;q^{r}e^{ix})=C_{1}(2k,q)(q^{r}e^{ix})^{(2k-1)}\times\sum_{j\in\mathbb{Z}}\sum_{\nu=0}^{2k-1}\frac{e^{(\nu-r)t}e^{-ix}-1}{-i(x+2\pi j)+(\nu-r)t}. \tag{20}$$

We have already specialized the more general result in [16] for our purposes and include a short proof for convenience of the reader.

Proof. By (14) and (15) the function

$$f(q;y):=C_{1}(2k,q)\prod_{\nu=0}^{2k-1}\frac{e^{(\nu-r)t}e^{-iy}-1}{-iy+(\nu-r)t}$$

is the Fourier transform of $M_{2k}(-\cdot)\,q^{r\cdot}$. Hence

$$M_{2k}(-v)\,q^{rv}=(2\pi)^{-1}\int f(q;y)\,e^{-ivy}\,dy.$$

By the Poisson summation formula this implies that the function

$$\sum_{j\in\mathbb{Z}}f(q;x+2\pi j)$$

has the same Fourier coefficients as $\sum M_{2k}(-v)\,q^{rv}e^{ivx}$. Therefore it must coincide with $(q^{r}e^{ix})^{(1-2k)}\,Q_{k,0}(q;q^{r}e^{ix})$. ∎

In the next chapter we need some properties of the zeros of the polynomials $Q_{k,0}(q;\cdot)$.

THEOREM 3 [16]. *The polynomial* $Q_{k,0}(q;\cdot)$ *has* $2k-2$ *negative and simple zeros*

$$\lambda_{2k-2}(q) < \cdots < \lambda_1(q) \tag{21}$$

which are strictly decreasing functions of q *mapping* $(0,\infty)$ *onto* $(-\infty,0)$. *In particular we have*

$$\lambda_k(1) < -1 < \lambda_{k-1}(1) \tag{22}$$

and

$$\lambda_v^{-1}(q) = \lambda_{2k-1-v}(q^{-1}). \tag{23}$$

Recently the polynomials $Q_{k,0}(q;z)$ have been investigated by Feng and Kozak [10] who obtained further nice properties of the zeros $\lambda_v(q)$.

3. BOUNDEDNESS OF THE PROJECTIONS $P_{k,r}$

As mentioned already in the Introduction, we have $\|P_{k,r}\|_\infty \sim |Q_{k,r}(q;-1)|^{-1}$. Applying Theorem 2 we obtain

$$Q_{k,r}(q;-1) = C_6(k,r,q)(-1)\sum_{j\in\mathbb{Z}}\prod_{v=0}^{2k-1}\frac{1}{i(\pi+2\pi j)+(r-v)t}, \tag{24}$$

where

$$\begin{aligned} C_6(k,r,q) &= C_5(k,r,q)\,C_1(2k,q)\prod_{v=0}^{2k-1}(q^{v-r}+1) \\ &= 2r!\,(2k-1-r)!\,t^{2k-1}\prod_{v=1}^{r}\frac{q^v+1}{q^v-1}\prod_{v=1}^{2k-1-r}\frac{q^v+1}{q^v-1}. \end{aligned}$$

Since $Q_{k,r}$ has real coefficients this shows that $Q_{k,r}(q;-1) = Q_{k,2k-1-r}(q;-1)$ and (9) implies

$$\|P_{k,r}\|_\infty \sim \|(A_{k,r})^{-1}\|_\infty = \|(A_{k,2k-1-r})^{-1}\|_\infty \sim \|P_{k,2k-1-r}\|_\infty. \tag{25}$$

THEOREM 4.

$$\lim_{q\to 1} Q_{k,r}(q;-1) = (2/\pi)^{2k}(-1)^{k-1}\sum_{j\in\mathbb{Z}}(1+2j)^{-2k}, \tag{26}$$

$$\lim_{q\to\infty} Q_{k,r}(q;-1) = (-1)^r\frac{2k-1-2r}{2k-1}. \tag{27}$$

The first part of the Theorem is obvious. To prove the second statement, we need the following Lemma.

LEMMA 5. *Let f be analytic in a neighborhood of the interval $[-1, 1]$. Then the Cauchy principal value of $f(z)/z$ can be approximated by Riemann sums, i.e.,*

$$\begin{aligned}\text{pr.v.} \int_{-1}^{1} f(z)/z\, dz &= \lim_{\epsilon \to 0} \left\{ \int_{-1}^{-\epsilon} f(z)/z\, dz + \int_{\epsilon}^{1} f(z)/z\, dz \right\} \\ &= \lim_{n \to \infty} \frac{1}{n} \sum_{\nu=1}^{n-1} \frac{n}{\nu} \left(f\left(\frac{\nu}{n}\right) - f\left(-\frac{\nu}{n}\right) \right). \qquad (28)\end{aligned}$$

The *proof* of the Lemma is a straightforward application of the mean value theorem and is therefore omitted.

Proof of (27). We first observe that $\lim_{q \to \infty} t^{1-2k} C_6(k, r, q) = 2r!\,(2k-1-r)!$.

Substituting this in (24) yields

$$\begin{aligned}&\lim_{q \to \infty} Q_{k,r}(q; -1) \\ &= -2r!\,(2k-1-r)! \lim_{t \to \infty} t^{-1} \sum_{j \in \mathbb{Z}} \prod_{\nu=0}^{2k-1} \frac{1}{i(\pi + 2\pi j)/t + (r - \nu)}.\end{aligned}$$

By the previous Lemma, the last limit equals

$$(1/2\pi)\, \text{pr.v.} \int_{-\infty}^{\infty} \prod_{\nu=0}^{2k-1} \frac{1}{ix + (r-\nu)}\, dx.$$

Applying the residue theorem [1] we obtain

$$\text{pr.v.} \int \cdots = 2\pi i \Big(\operatorname*{Res}_{\operatorname{Im} z > 0} \cdots + \tfrac{1}{2} \operatorname*{Res}_{z=0} \cdots \Big).$$

Since

$$\begin{aligned}&\operatorname*{Res}_{x = i\mu} \prod_{\nu=0}^{2k-1} \frac{1}{ix + (r-\nu)} \\ &= -i \prod_{\substack{\nu=0 \\ r-\nu \neq \mu}}^{2k-1} \frac{1}{-\mu + r - \nu} \\ &= -i(-1)^{(2k-1-r+\mu)} \{(r-\mu)!\,(2k-1-r+\mu)!\}^{-1}\end{aligned}$$

we finally end up with

$$\lim_{q\to\infty} Q_{k,r}(q;-1) = 2r!\,(2k-1-r)!\,(-1)^r$$

$$\times \sum_{\mu=1}^{r} (-1)^{\mu}\,\{(r-\mu)!\,(2k-1-r+\mu)!\}^{-1} + (-1)^r$$

$$= 2\binom{2k-1}{r}^{-1}(-1)^r\left\{\sum_{\mu=1}^{r}(-1)^{\mu}\binom{2k-1}{r-\mu}\right\} + (-1)^r.$$

Using the recurrence relation

$$\binom{2k-1}{r-\mu} = \binom{2k-2}{r-\mu-1} + \binom{2k-2}{r-\mu}$$

we obtain

$$= 2\binom{2k-1}{r}^{-1}(-1)^r\left\{-\binom{2k-2}{r-1} + (-1)^r\binom{2k-2}{r-r-1}\right\} + (-1)^r$$

$$= (-1)^r\left\{-\frac{2r}{2k-1}+1\right\}. \quad \blacksquare$$

Remark. C. de Boor independently proved (27) starting with the representation of $Q_{k,0}$ as a divided difference

$$Q_{k,0}(q;z) = t^{-2k+2}[0,\dots,2k-1]\,\frac{1}{q^{\cdot}-z}$$

obtained by Micchelli [16, Theorem 2.1]. It follows from (19′) that

$$Q_{k,r}(q;-1) = Q_{k,0}(q;-q^r)/Q_{k,0}(q;q^r).$$

Evaluating the corresponding divided differences leads, without computing an infinite series, to the limit as a difference of two factorial terms.

Theorem 4 shows that for a uniform sequence of knots the L_∞-norm of $(A_{k,r})^{-1}$ grows much more rapidly with the order k than it does for large values of the local mesh ratio.

THEOREM 5. *For $r = k-1, k$ we have*

$$\|P_{k,r}\|_\infty \sim \|(A_{k,r})^{-1}\|_\infty = |Q_{k,r}(q;-1)|^{-1}$$

$$= \left\{C_6(k,k,q)\sum_{j\in\mathbb{Z}}\prod_{\nu=1}^{k}\frac{1}{(\pi+2\pi j)^2+(\nu t)^2}\right\}^{-1} \tag{29}$$

and

$$\lim_{q\to 1} \|(A_{k,r})^{-1}\|_\infty = (\pi/2)^{2k} \left\{ \sum_{j\in\mathbb{Z}} (1+2j)^{-2k} \right\}^{-1}, \tag{30}$$

$$\lim_{q\to\infty} \|(A_{k,r})^{-1}\|_\infty = 2k-1, \tag{31}$$

i.e., the L_∞-norms of the projections $P_{k,k-1}$ and $P_{k,k}$ can be bounded uniformly with respect to $q \in (1, \infty)$.

Proof. We only have to check the formula for the norm of $(A_{k,k})^{-1}$. By (9) and (24) we have

$$(\|(A_{k,k})^{-1}\|_\infty)^{-1} = C_6 \left| \sum_{j\in\mathbb{Z}} a_j \frac{1}{iy_j(iy_j+kt)} \right|,$$

where $y_j = \pi + 2\pi j$ and

$$a_j = \prod_{\nu=1}^{k-1} \frac{1}{y_j^2 + (\nu t)^2}.$$

Since

$$\frac{1}{iy_j(iy_j+kt)} + \frac{1}{-iy_j(-iy_j+kt)} = \frac{-2}{y_j^2+(kt)^2}$$

we finally obtain

$$\left| \sum_{j\in\mathbb{Z}} a_j \frac{1}{iy_j(iy_j+kt)} \right| = 2 \sum_{j\in\mathbb{N}_0} a_j \frac{1}{y_j^2+(kt)^2}. \quad \blacksquare$$

Remark. Recently Feng [11] has shown that $2k-1$ is in fact a lower bound for the L_∞-norm of $(A_{k,k})^{-1}$. Therefore one might conjecture that $|Q_{k,k}(q;-1)|^{-1}$ is a monotone decreasing function of q.

COROLLARY 2. *The eigenvalues λ_ν (21) of the interpolation problem, i.e., the zeros of the polynomials $Q_{k,0}(q;\cdot)$, satisfy*

$$\left. \begin{array}{ll} |\lambda_\nu(q)| < q^\nu, & \nu = 1,\ldots,k-1 \\ |\lambda_\nu(q)| > q^\nu, & \nu = k,\ldots,2k-2 \end{array} \right\}, \quad q \in (1,\infty). \tag{32}$$

Proof. By (22) the inequalities are satisfied for $q=1$. If they were violated, e.g., for $\nu = k$, then there would exist $q_0 \in (1,\infty)$ with $\lambda_k(q_0) = -q_0^k$. But this would imply that $Q_{k,0}(q_0; -q_0^k) = Q_{k,k}(q;-1) = 0$ which is impossible by Theorem 5 (29).

The case $\nu = k-1$ is treated similarly.

For the remaining values of ν the inequalities are proved using the interlacing properties of the eigenvalues $\lambda_\nu(q)$ for fixed q and increasing order k [12, p. 223]. ∎

COROLLARY 3. *For $r \neq k-1, k$ there exists at least one value of $q \in (1, \infty)$ such that $\|P_{k,r}\|_\infty$ is not bounded.*

Proof. We may assume $r < k-1$ and have to show that $Q_{k,r}(q; -1) \sim Q_{k,0}(q; -q^r)$ vanishes for some $q_0 \in (1, \infty)$. For $r = k-2$ this follows because by Theorem 4 $Q_{k,k-2}(\cdot; -1)$ changes sign in the interval $(1, \infty)$, i.e., $\lambda_\nu(q_0) = -q_0^{k-2}$. By Corollary 2 we have $\nu \leq k-1$. Since $\lambda_\nu(q) > \lambda_{k-1}(q)$ there must also exist q_1 with $\lambda_{k-1}(q_1) = -q_1^{k-2}$. But this implies that the equation $\lambda_{k-1}(q) = -q^r$ has a solution for any $r < k-2$, too, which is equivalent to the existence of a zero of the characteristic polynomial $Q_{k,r}(q; -1)$. ∎

We conjecture that there are exactly $k-1-r$ values q_ν for which $\|P_{k,r}\|_\infty \sim \|P_{k,2k-1-r}\|_\infty$, $0 < r < k-1$, is not bounded. To this end it suffices to show that the functions

$$|\lambda_\nu(q)|/q^{\nu-1}, \qquad \nu = 2, \ldots, k-1$$

are strictly increasing for $q \in (1, \infty)$ (cf. also [12, Theorem 3.7c]).

4. EXTENSION TO L_p

The following result is due to C. de Boor (unpublished manuscript).

THEOREM 6. *Let $A_{\nu\mu} = (a_{\mu-\nu})$ be a totally positive constant band matrix and denote by*

$$Q(z) = \sum_\nu a_\nu z^\nu$$

the characteristic polynomial. Then A is invertible on l_p iff $Q(-1) \neq 0$. Moreover we have

$$\|A^{-1}: l_p \to l_p\| = |Q(-1)|^{-1}, \qquad 1 \leq p \leq \infty, \tag{33}$$

i.e., the norm of the inverse does not depend on p.

Proof. The first statement reduces to the case $p = \infty$ since, by a result of Demko [9], for band matrices invertibility on l_p is equivalent to invertibility on l_∞ for any $p \in [1, \infty]$. To prove (33) we may assume that $Q(-1) \neq 0$, i.e., A^{-1} exists. By the assumptions on A we have $(A^{-1})_{\nu\mu} = b_{\mu-\nu}$ and $\operatorname{sgn} b_\nu = \pm(-1)^\nu$ [15]. This already implies

$$\|A^{-1}\|_1 = \|A^{-1}\|_\infty = |Q(-1)|^{-1}.$$

Interpolating between l_1 and l_∞ we obtain [2]

$$\|A^{-1}\|_p \leqslant |Q(-1)|^{-1}.$$

We shall prove equality for $p = 2$, too. Then the Theorem follows by an interpolation argument. Suppose, e.g., $\|A^{-1}\|_p < |Q(-1)|^{-1}$, $p < 2$. Interpolating between l_p and l_∞ leads to the contradiction

$$\|A^{-1}\|_2 \leqslant \|A^{-1}\|_p^\theta \, \|A^{-1}\|_\infty^{1-\theta} < |Q(-1)|^{-1}.$$

To prove that $\|A^{-1}\|_2^{-1} = |Q(-1)|$ we first observe that the checkerboard pattern of A^{-1} implies

$$|Q(-1)| = \operatorname*{Min}_{-\pi \leqslant x \leqslant \pi} |Q(e^{ix})|.$$

Using the Bessel identity we obtain

$$\begin{aligned}
\|A^{-1}\|_2^{-1} &= \inf_{\|f\|_{l_2} = 1} \|Af\|_{l_2} \\
&= \inf_{\|F = \sum_\nu f_\nu e^{i\nu\cdot}\|_2 = 1} \Big\| \sum_\nu a_{\mu-\nu} f_\mu e^{i\nu\cdot} \Big\|_2 \\
&= \inf_{\|F\|_2 = 1} \Big\| \Big(\sum_\mu f_\mu e^{-i\mu\cdot}\Big)\Big(\sum_\nu a_\nu e^{i\nu\cdot}\Big) \Big\|_2 \\
&= \inf_{-\pi \leqslant x \leqslant \pi} |Q(e^{ix})|. \qquad \blacksquare
\end{aligned}$$

For $1 \leqslant p \leqslant \infty$, $0 < r < 2k$, we define the matrices

$$(A_{k,r}^{(p)})_{\nu\mu} = C_7(k,r,q,p)\, q^{\nu/p} (N_r, N_{2k-r}(q^{\nu-\mu}\cdot))\, q^{-\mu/p}, \tag{34}$$

where $C_7(k,r,q,p) = (1/(q^r - 1))^{1/p'} \, (1/(q^{2k-r} - 1))^{1/p}$ and the corresponding characteristic polynomials

$$Q_{k,r}^{(p)}(q; z) = C_7(k,r,q,p)\, z^{2k-1-r} \sum_\nu (N_r, N_{2k-r}(q^{-\nu}\cdot))\, q^{-\nu/p} z^\nu. \tag{35}$$

Extending the result in [4, p. 538] to L_p, we obtain by (3) and Theorem 6,

$$\|P_{k,r} : L_p \to L_p\| \sim \|(A_{k,r}^{(p)})^{-1}\|_p = |Q_{k,r}^{(p)}(q; -1)|^{-1}. \tag{36}$$

LEMMA 6. *$P_{k,r}$, $0 < r < 2k$, is bounded with respect to the L_p-norm iff $Q_{k,0}(q; -q^{r-1/p})$ does not vanish. Moreover, $P_{k,r}$ is the adjoint of $P_{k,2k-r}$, which implies*

$$\|P_{k,r}\|_p = \|P_{k,2k-r}\|_{p'}. \tag{37}$$

Proof. The first part of the Lemma follows directly from Corollary 1 and the definition of the polynomial $Q_{k,r}^{(p)}(q; z)$. By Lemma 4 we have, for $s \in \mathbb{R}$,

$$\begin{aligned} Q_{k,0}(q; -q^s) &= C \sum_{\nu} M_{2k}(-\nu)(-q^s)^{\nu} \\ &= C \sum_{\nu} M_{2k}(2k+\nu)\, q^{(2k-1)(-\nu)}(-q^s)^{\nu} \\ &= C \sum_{\mu} M_{2k}(-\mu)\, q^{(2k-1)(\mu+2k)}(-q^s)^{-(\mu+2k)} \\ &= C \sum_{\mu} M_{2k}(-\mu)(-q^{(2k-1-s)})^{\mu} \\ &= CQ_{k,0}(q; -q^{2k-1-s}). \end{aligned}$$

Here C is a nonzero constant depending on k, r, p, q.

Hence by the first part of the Lemma the existence of the projection $P_{k,r}$: $L_p \to L_p$ is equivalent to the existence of the projection $P_{k,2k-1-r}: L_{p'} \to L_{p'}$. It is now easy to check that $P_{k,r}^* = P_{k,2k-1-r}$. ∎

THEOREM 7. *For a geometric mesh (5) the L_2-projection $P_{k,k}: L_2 \to S_k$ is bounded with respect to every L_p-norm uniformly in $p \in [1, \infty]$ and $q \in [1, \infty)$. For $r \neq k$, $1 \leqslant p \leqslant \infty$*

$$P_{k,k-1}: L_\infty \to L_\infty \cap S_{k-1}$$

and

$$P_{k,k+1}:\ L_1 \to L_1 \cap S_{k-1}$$

are the only projections which are bounded for every $q \in (1, \infty)$.

Proof. The first part of the Theorem follows from Lemma 6 and Theorem 5 by the usual interpolation argument.

We shall show that for $r = k-1$, $p < \infty$ or $r < k-1$, $\|P_{k,r}\|_p$ is not bounded for some $q \in (1, \infty)$. By Lemma 6 it suffices to show that $Q_{k,0}(q; -q^s)$ changes sign on the interval $(1, \infty)$ for $s < k-1$. Arguing as in the proof of Corollary 3 we may suppose $k-2 < s < k-1$. By (20) the sign changes of $Q_{k,0}(q; -q^s)$, $q \in (1, \infty)$ correspond to sign changes of the function

$$R(s, t) := t^{2k-1} \sum_{j \in \mathbb{Z}} \prod_{v=0}^{2k-1} \frac{1}{i(\pi + 2\pi j) + (s - v)\, t}.$$

Following the proof of Theorem 4 we obtain

$$\lim_{t \to 0} \operatorname{sgn} R(s, t) = (-1)^k.$$

$$\lim_{t \to \infty} R(s, t) =: R(s, \infty) = \frac{1}{2\pi} \text{pr.v.} \int_{\mathbb{R}} \prod_{v=0}^{2k-1} \frac{1}{ix + (s - v)}.$$

With $f(x) := \prod_{v=0}^{2k-1} 1/(ix - v)$ we obtain for $s \in (k-2, k-1)$

$$\begin{aligned} R(s, \infty) &= i \sum_{v=0}^{k-2} \operatorname*{Res}_{x = i(s-v)} f(x - is) \\ &= i \sum_{v=0}^{k-2} \operatorname*{Res}_{x = i(k-1-v)} f(x - i(k-1)). \end{aligned}$$

Comparing this with the proof of Theorem 4 we get

$$\begin{aligned} R(s, \infty) &= R(k-1, \infty) - \frac{i}{2} \operatorname*{Res}_{x=0} f(x - (k-1)) \\ &= \left\{ -(2(k-1)!\, k!)^{-1} (-1)^{k-1} \frac{1}{2k-1} \right\} \\ &\quad - \left\{ \frac{i}{2} (-i)(-1)^k ((k-1)!\, k!)^{-1} \right\} \\ &= \frac{1}{2} ((k-1)!\, k!)^{-1} (-1)^k \left\{ \frac{1}{2k-1} - 1 \right\}. \end{aligned}$$

This implies that

$$\operatorname{sgn} R(s, \infty) = (-1)^{k-1} = -\lim_{t \to 0} \operatorname{sgn} R(s, t),$$

i.e., $Q_{k,0}(q; -q^s)$ changes sign on $(1, \infty)$ which completes the proof of the Theorem. ∎

In particular it follows from Theorem 7 that $P_{k,k-1}$ cannot be defined on L_2 for some $q \in (1, \infty)$ whereas $P_{k,k-1}$ is bounded on L_∞ uniformly in q. This is quite unexpected from the definition of the projections $P_{k,r}$.

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